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ON THE STABILITY OF ELASTIC BODIES WITH RANDOM INHOMOGENEITIES UNDER FINITE DEFORMATIONS

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Stability of stochastically inhomogeneous, compressible elastic bodies with respect to small, as well as to finite perturbations, is studied in three-dimensional formulation. The bodies are under deterministic external loads and experience finite subcritical deformations.

The stability of elastic bodies with random inhomogeneities was studied for the case of small, subcritical deformations in [1]. The basic relations for a stochastically inhomogeneous compressible hyperelastic body can be obtained from the relations for compressible hyperelastic media given in [2].

1. We write the equations of state as follows:

$$s_i^j = L_i^j\Phi, \quad L_i^j = \delta_i^j \frac{\partial}{\partial A_1} + 2\varepsilon_i^j \frac{\partial}{\partial A_2} + 3p_i^j \frac{\partial}{\partial A_3}, \quad p_i^j = \varepsilon_i^n \varepsilon_n^j \tag{1.1}$$
$$A_1 = \varepsilon_n^n, \ A_2 = \varepsilon_n^l \varepsilon_l^n, \ A_3 = \varepsilon_n^m \varepsilon_m^l \varepsilon_l^n, \ \Phi = \Phi(A_1, A_2, A_3, c_p)$$

Here c_p denote the parameters of the medium $(p = 1, 2, \ldots, \Pi)$ depending on the spatial coordinates in a random manner; A_i are the algebraic invariants; ε_i^j are the deformation tensor components and s_i^j are the generalized stress tensor components. The covariant components of the Green deformation tensor are written in the form

$$2\varepsilon_{ij} = G_{ij} - g_{ij}, \ G_{ij} = c_i^n c_j^l g_{nl}, \ c_i^n = \delta_i^n + \nabla_i u^n \tag{1.2}$$

while the equations of equilibrium and the boundary conditions in terms of the stresses are

$$g^{nl}\nabla_i (s_n^i c_l^m) + \rho X^m = 0, \ g^{nl} s_n^i c_l^m N_i = P^m \tag{1.3}$$

where g^{ij} is the metric tensor of the Lagrangian deterministic reference frame of the initial state, P^m are the components of the deterministic surface forces and N_i are the deterministic unit vectors of the normal to the surface of the body prior to deformation.

Since the parameters of the medium in (1.1) depend on the spatial coordinates in a random manner, it follows that the field quantities in the relations (1.1) — (1.3) will also be random functions of the spatial coordinates.

Let us assume that the parameters of the medium depend on the homogeneous, isotropic random function

$$c_p = \langle c_p \rangle f = \langle c_p \rangle (1 + f') = \langle c_p \rangle + c_p' \quad (p = 1, 2, \ldots, \Pi)$$

Here and henceforth $\langle x\rangle$ denotes the mathematical expectation of the quantity x, and x' denotes its fluctuation.

Consecutive application of the method of statistical linearization yields the mathematical expectations of the field functions, and their fluctuations, in the form

$$c_p = \langle c_p\rangle + c_p', \quad c_p' = c_{p3}f', \quad \varepsilon_i^j = \langle \varepsilon_i^j\rangle + \varepsilon_i^{j'}, \quad \varepsilon_i^{j'} = \varepsilon_{i3}^{\ j} f' \qquad (1.4)$$
$$s_i^j = \langle s_i^j\rangle + s_i^{j'}, \quad s_i^{j'} = s_{i3}^{\ j} f', \quad \langle s_i^j\rangle = s_{i1}^{\ j} + s_{i2}^{\ j}\langle f'^2\rangle$$
$$\langle c_p\rangle = c_{p1} + c_{p2}\langle f'^2\rangle, \quad \langle \varepsilon_i^j\rangle = \varepsilon_{i1}^{\ j} + \varepsilon_{i2}^{\ j}\langle f'^2\rangle$$

Clearly, we have $c_{p3} = c_{p1} = \langle c_p\rangle$, $c_{p2} = 0$.

Using (1.4), we can obtain from (1.1) – (1.3) an averaged and a fluctuational system of equations. The assumption that the fluctuations are small, enables us to linearize (1.1) – (1.3) and obtain these systems. Thus, any function $B(x)$ can be expanded into a Taylor series

$$B(x) = B(x_1 + x_2\langle f'^2\rangle + x_3 f') = B(x_1) + \frac{\partial B}{\partial x_1} x_3 f' + \frac{1}{2}\frac{\partial^2 B}{\partial x_1^2} x_3 x_3 f'^2 + \frac{\partial B}{\partial x_1} x_2 \langle f'^2\rangle$$

and from this we have

$$\langle B(x)\rangle = B(x_1) + \left(\frac{1}{2}\frac{\partial^2 B}{\partial x_1^2} x_3 x_3 + \frac{\partial B}{\partial x_1} x_2\right)\langle f'^2\rangle, \quad B'(x) = \frac{\partial B}{\partial x_1} x_3 f' \qquad (1.5)$$

Let us call the term $B(x_1)$ system 1, the term preceding $\langle f'^2\rangle$ system 2, and the term preceding f' system 3. Thus, to obtain the averaged system of equations we must add to the equations of system 1, the equations of system 2 multiplied by $\langle f'^2\rangle$. The fluctuational system is the same as system 3 multiplied by f'.

For (1.1) the system 1 – 3 of equations have the form

$$s_{ik}^{\ j} = L_{i\alpha}^{\ j}\Phi_\beta, \quad L_{ik}^{\ j} = \delta_{ik}^{\ j}\frac{\partial}{\partial A_{11}} + 2\varepsilon_{ik}^{\ j}\frac{\partial}{\partial A_{21}} + 3p_{ik}^{\ j}\frac{\partial}{\partial A_{31}}$$
$$\delta_{ik}^{\ j} = \delta_i^j, \quad k = 1; \quad \delta_{ik}^{\ j} = 0, \quad k = 2, 3 \qquad (1.6)$$
$$p_{ik}^j = \varepsilon_{i\alpha}^n \varepsilon_{n\beta}^j, \quad A_{1k} = \varepsilon_{nk}^n, \quad A_{2k} = \varepsilon_{n\alpha}^l \varepsilon_{l\beta}^n, \quad A_{3k} = \varepsilon_{n\alpha}^l \varepsilon_{l\beta}^m \varepsilon_{m\gamma}^n$$
$$\Phi_k = M_k\Phi_1, \ \Phi_1 = \Phi(A_{11}, A_{21}, A_{31}, \langle c_p\rangle), \ k = 1, 2, 3$$
$$M_1 = 1, \quad M_2 = A_{n2}\frac{\partial}{\partial A_{n1}} + \frac{1}{2}A_{l3}A_{n3}\frac{\partial^2}{\partial A_{l1}\partial A_{n1}} + A_{l3}c_{p3}\frac{\partial^2}{\partial A_{l1}\partial c_{p1}} + \frac{1}{2}c_{q3}c_{p3}\frac{\partial^2}{\partial c_{q1}\partial c_{p1}}, \quad M_3 = A_{n3}\frac{\partial}{\partial A_{n1}} + c_{p3}\frac{\partial}{\partial c_{p1}}$$

The potential Φ_1 coincides with any potential of the determinate problem Φ, provided that the last lower index in every term is 1.

For the geometrical relations we have

$$2\varepsilon_{ik}^j = G_{ink}g^{nj}, \quad G_{ijk} = c_{i\alpha}^n c_{j\beta}^l g_{nl}, \quad c_{ik}^n = \delta_{ik}^n + \nabla_i u_k^n \qquad (1.7)$$

and for the equations of equilibrium and the boundary conditions (1.3) we have

$$g^{ln}\nabla_i(s_{n\alpha}^i c_{l\beta}^m) + X_k^{\ m} = 0, \quad g^{ln}s_{l\alpha}^i c_{n\beta}^m N_i = P_k^{\ m}; \quad P_k^{\ m} = \begin{cases} P^m, & k = 1 \\ 0, & k = 2, \end{cases} \qquad (1.8)$$

Here and henceforth the index k will indicate whether the relations (1.6) — (1.8) belong to the system 1 , 2 or 3 . Summation over the Greek letter indices is implicit and obeys the rule

$$\sum_{\alpha,\beta,\gamma} a_\alpha b_\beta c_\gamma = \begin{cases} a_1b_1c_1, & k=1 \\ a_1b_1c_1+a_1b_2c_1+a_2b_1c_1+a_2b_1c_1+a_1b_3c_3+a_3b_1c_3+a_3b_3c_1, & k=2 \\ a_1b_1c_3+a_1b_3c_1+a_3b_1c_1, & k=3 \end{cases}$$

It is clear that the elongation λ_i is also a random quantity which can be written, according to (1.4), in the form

$$\lambda_i = \lambda_{i1} + \lambda_{i2}\langle f'^2\rangle + \lambda_{i3} f' \tag{1.9}$$

where λ_{ik} represent certain, non-random quantities.

When the initial state in macro-homogeneous,

$$s_{ik}{}^j = \mathrm{const}_{ik}\delta_i{}^j \tag{1.10}$$

we obtain

$$c_{ik}^j = \delta_{ik}^j + \nabla_i u_k{}^j = \lambda_{ik}\delta_i{}^j, \quad \lambda_{ik} = \mathrm{const}_{ik}; \quad 2e_{ik}{}^j = (\lambda_{i\alpha}\lambda_{j\beta} - \delta_{ik}^j)\,\delta_i{}^j \tag{1.11}$$

2. Let us investigate the stability with respect to small perturbations. We denote the components of the field functions of perturbed state by the index plus, those of the nonperturbed state by index zero, and leave the components of the perturbations without any indices

$$\begin{gathered} s^{+}{}_i^j = s_i{}^{\circ j} + s_{i2}{}^\circ\langle f'^2\rangle + s_{i3}{}^{\circ j} + s_{i1}{}^j + s_{i2}^j\langle f'^2\rangle + s_{i3}{}^j f' \\ u_i{}^+ = u_{i1}{}^\circ + u_{i2}{}^\circ\langle f'^2\rangle + u_{i3}{}^\circ f' + u_{i1} + u_{i2}\langle f'^2\rangle + u_{i3} f', \ldots \end{gathered} \tag{2.1}$$

The quantities with index zero can be computed using the formulas of Sect. 1, by putting a zero index everywhere. Systems linear with respect to perturbations are obtained by linearizing (1.6) —(1.8). The equations of state (1.6) yield

$$\begin{gathered} s_{ik}{}^j = L_{i\alpha}{}^j\Phi_\beta + L_{i\alpha}{}^j\Phi_\beta{}^\circ, \quad L_{ik}{}^j = 2\varepsilon_{ik}{}^j\frac{\partial}{\partial A_{21}{}^\circ} + 3p_{ik}{}^j\frac{\partial}{\partial A_{31}{}^\circ} \\ p_{ik}^j = \varepsilon_{i\alpha}^n\overset{\circ}{\varepsilon}{}_{n\beta}^j + \overset{\circ}{\varepsilon}{}_{i\alpha}^n\varepsilon_{n\beta}^j, \quad A_{1k} = \varepsilon_{nk}^n, \quad A_{2k} = 2\overset{\circ}{\varepsilon}{}_{n\alpha}^l\varepsilon_{l\beta}^n \\ A_{3k} = 3\overset{\circ}{\varepsilon}{}_{n\alpha}^l\overset{\circ}{\varepsilon}{}_{l\beta}^m\varepsilon_{m\gamma}^n, \quad \Phi_k = M_k\Phi^\circ, M_k = M_\alpha{}^\circ A_{l\beta}\frac{\partial}{\partial \overset{\circ}{A}_{l\beta}} \end{gathered} \tag{2.2}$$

The geometrical relations (1.7), the equations of equilibrium (with the mass forces neglected) and the boundary conditions (1.8) together yield

$$2\varepsilon_{ik}^j = G_{ink}g^{\circ nj}, \quad G_{ijk} = (\overset{\circ}{c}{}_{i\alpha}^l c_{j\beta}^n + c_{i\alpha}^l\overset{\circ}{c}{}_{j\beta}^n)\,\overset{\circ}{g}_{ln}, \quad c_{ik}^n = \nabla_i u_k{}^n \tag{2.3}$$

$$g^{\circ ln}\nabla_i(\overset{\circ}{s}{}_{n\alpha}^i c_{l\beta}^m + s_{n\alpha}^i\overset{\circ}{c}{}_{l\beta}^m) = \rho\ddot{u}_k{}^m, \quad g^{\circ ln}(\overset{\circ}{s}{}_{l\alpha}^i c_{n\beta}^m + s_{l\alpha}^i\overset{\circ}{c}{}_{n\beta}^m)\,N_i{}^\circ = P_k{}^m \tag{2.4}$$

In the general case the linearized equations of state (2.2) can be written for a compressible body in the form

$$s_{nk}{}^l = \lambda_{nm\cdot\alpha}^{i\cdot\cdot l}\nabla_l u_\beta{}^m. \tag{2.5}$$

Direct substitution confirms that the quantities $\lambda^{i\cdot\cdot l}_{nm\cdot\alpha}$ represent the components of a fourth rank tensor satisfying the conditions

$$\lambda^{i\cdot\cdot l}_{nm\cdot k} = \lambda^{n\cdot\cdot l}_{\;\;m\cdot k}, \quad \lambda^{i\cdot\cdot m}_{nl\cdot k} \neq \lambda^{i\cdot\cdot l}_{nm\cdot k}, \quad \lambda^{m\cdot\cdot i}_{ln\cdot k} \neq \lambda^{i\cdot\cdot m}_{nl\cdot k}$$

The relations (2.5) are written in a general form, and the tensor $\lambda^{i\cdot l}_{nm\cdot k}$ is not given in full because of its bulk.

Let us inspect the simplifications which arise in the case of a macrohomogeneous initial state (1.10), (1.11). The linearized equations of state can be written in the orthogonal coordinate system in the form

$$s^{j}_{ik} = \delta_i^{\;j} a_{in\alpha}\lambda_{n\beta}\nabla_n u_\gamma^{\;n} + (1-\delta_i^{\;j})\,\mu_{ij\alpha}\,(\lambda_{j\beta}\nabla_i u_\gamma^{\;j} + g^{jj} g_{ji}\lambda_{j\beta}\nabla_j u_\gamma^{\;i}) \qquad (2.6)$$

Then the coefficients $a_{in\alpha}$ and $\mu_{ij\alpha}$ of the equation of state (2.6) can be computed for the Murnaghan-type potential [2]

$$\Phi = \frac{1}{2}\lambda A_1^2 + \mu A_2 + \frac{a}{3}A_1^3 + bA_1A_2 + \frac{c}{3}A_3$$

from the formulas

$$a_{ink} = 2\,[{}^1\!/_2\lambda_k + a_\alpha A_{1\alpha} + b_\alpha(\varepsilon^i_{i\alpha} + \varepsilon^n_{n\alpha}) + \mu_{iik}\delta_i^{\;n}]$$
$$\mu_{ijk} = \mu_k + b_\alpha A_{1\alpha} + {}^1\!/_2 c_\alpha(\varepsilon^i_{i\alpha} + \varepsilon^j_{j\alpha})$$

Substituting the equations of state (2.5) into the equations of motion and boundary conditions (2.4) at the free surface, we arrive at the following system of homogeneous differential equations in displacements, with the boundary conditions at the free surface:

$$L_{mn\alpha}u_{n\beta} = M_k,\; D_{mn\alpha}u_{n\beta} = M_k,\; M_k = 0 \qquad (2.7)$$

Here L_{mnk} and D_{mnk} are the differential operators of the second and first order, respectively, and their form depends on the actual formulation of the problem.

Systems of equations for the mathematical expectations and fluctuations can be written, according to Sect. 1, in the form

$$L_{mn1}u_{n1} + \langle f'^2\rangle\,(L_{mn1}u_{n2} + L_{mn2}u_{n1} + L_{mn3}u_{n3}) = 0$$
$$L_{mn1}u_{n3} + L_{mn3}u_{n1} = 0$$
$$D_{mn1}u_{n1} + \langle f'^2\rangle\,(D_{mn1}u_{n2} + D_{mn2}u_{n1} + D_{mn3}u_{n3})\,|_{x\in S} = 0$$
$$D_{mn1}u_{n3} + D_{mn3}u_{n1}\,|_{x\in S} = 0$$

and this yields, by virtue of the arbitrariness of $\langle f'^2\rangle$, the boundary value problem (2.7). The critical load is found either from the condition of existence of a nontrivial solution of the problem (2.7), in the case of static problems, or from the condition that the displacements do not increase with time, in the case of the dynamic problems. We note that the solution of the boundary value problem with index 1 represents a solution of a determinate boundary value problem with parameters $c_p = c_{p1} = \langle c_p\rangle$.

Thus, as a result of solving the determinate boundary value problem, we obtain the critical values of the quantities with index 1: the critical force $P_1{}^m$, elongation λ_{n1}, deformation $\varepsilon_{n1}{}^{on}$ and stress $s_{n1}{}^{on}$. Knowing the critical characteristic values with index 1, we can recover the quantities with indices 2 and 3 from the second and third boundary value problem for the nonperturbed state (1.6) – (1.8). The values of the field functions under the action of the force $P_1{}^m$ can be obtained from (1.4) and (1.9).

3. Let us now extend the arguments given above to the case when finite perturbations are imposed on the basic state of the body as described by the relations (1.6) – (1.8). In this case the right hand sides of the relations (2.2) and (2.3) must be supplemented by the following corresponding terms:

$$s_{ik}^{j} \to L_{i\alpha}^{j}\Phi_{\beta}, \quad p_{ik}^{j} \to \varepsilon_{i\alpha}^{n}\varepsilon_{n\beta}^{j}, \quad A_{2k} \to 2\varepsilon_{n\alpha}^{l}\varepsilon_{l\beta}^{n}$$
$$A_{3k} \to 3\overset{\circ}{\varepsilon}{}_{n\alpha}^{l}\varepsilon_{l\beta}^{m}\varepsilon_{m\gamma}^{n} + \varepsilon_{n\alpha}^{l}\varepsilon_{l\beta}^{m}\varepsilon_{m\gamma}^{n}, \quad G_{ijk} \to c_{j\alpha}^{l}c_{j\beta}^{n}\overset{\circ}{g}_{ln}$$

The relations appearing in (2.2) and (2.3) which are not given here, remain unchanged.

Expansion of the function Φ_k into a Taylor series yields the following expression:

$$\Phi_k = \sum_{l,p,s}^{l+p+s=n} \frac{\partial^n \Phi^\circ}{n!\,\partial \overset{\circ}{A}{}_{1\alpha}^{l}\partial \overset{\circ}{A}{}_{2\beta}^{p}\partial \overset{\circ}{A}{}_{3\gamma}^{s}} A_{1\alpha}^{l}A_{2\beta}^{p}A_{3\gamma}^{s} \qquad (3.1)$$
$$(l + p + s = n;\ l,\ p,\ s = 0,1,\ldots;\ n = 1,2,\ldots)$$

The explicit form of (3.1) is governed by the actual form of the elastic potential. Equations of equilibrium and boundary conditions are obtained from (2.4) by adding the terms

$$\overset{\circ}{g}{}^{ln}\nabla_i s_{n\alpha}^{i} c_{l\beta}^{m}, \quad \overset{\circ}{g}{}^{ln} s_{n\alpha}^{i} c_{n\beta}^{m} \overset{\circ}{N}_i$$

to the left hand sides of the corresponding expressions.

We shall write the solution of the resulting nonlinear boundary value problem in the form of a series ($y_l(t)$, are functions of time)

$$u_m = y_l(t)\,\varphi_m{}^l(x_i) \quad (m = 1, 2, 3;\ l = 1, 2, \ldots) \qquad (3.2)$$

We choose the forms of flexures with respect to small perturbations as the basis functions $\varphi_m{}^l$ satisfying the geometrical boundary conditions. We also assume that the condition of completeness of the system of functions $\varphi_m{}^l$ [3] represents a sufficient condition for the convergence of the series (3.2).

We write the relation (3.2), by virtue of (1.5), in the form

$$u_{mk} = y_{l\alpha}(t)\ \varphi_{m\beta}^{l}(x_i) \qquad (3.3)$$

Constructing the variational equations of the Bubnov – Galerkin method [4] corresponding to the nonlinear boundary value problem and taking into account (3.3) and the smallness of the fluctuations, we obtain

$$L_{m1}{}^l(y_1) + \langle f'^2\rangle L_{m2}{}^l(y_1, y_2, y_3) = 0, \quad L_{m3}{}^l(y_1, y_3) = 0$$
$$L_{mk}^{l}(y_1, y_2, y_3) = A_{m\alpha}^{l}y_\beta^{\cdot\cdot} + B_{m\alpha}^{l}y_\beta^{\cdot} + F_{m\alpha}^{l}y_\beta + E_{m\alpha}^{l}y_\beta y_\gamma + \ldots$$

This in turn yields, by virtue of the arbitrariness of $\langle f'^2 \rangle$, a system of nonlinear, ordinary differential equations. At the same time, the solution of the system of equations with index 1, i.e.

$$L_{m1}{}^l y_1 = 0; \; l = 1, 2, \ldots; \; m = 1, 2, 3 \tag{3.4}$$

corresponds to the solution of the determinate system of nonlinear, ordinary differential equations. Thus the problem of stability of the basic state has been reduced to the problem of stability of the zero solution of (3.4).

It is clear that, when the function

$$V = {}^1/_2 \, A_{11} y_1{}^{\cdot} y_1{}^{\cdot} + {}^1/_2 \, B_{11} y_1 y_1 + {}^1/_3 D_{11} y_1 y_1 y_1 + \ldots$$

is positive, then it represents a Liapunov function for the operator equations (3.4), since by virtue of the system its derivative is non-positive [5]. Consequently the condition of positiveness of the function will represent a sufficient condition for the stability of the zero solution of the system (3.4). The quantities with indices 2 and 3 can be recovered in the same manner.

In conclusion we note that the proposed approach enables us to obtain the mean values of the critical elongations and hence of the stresses, deformations and displacements, depending on the order of dispersion of the inhomogeneity. At the same time, the value of the critical force coincides, in the mean, with its value in the zero approximation.

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